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A continuum damage model for transverse matrix cracking in laminated fibre-reinforced composites

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In this paper, the effects of damage in the form of transverse matrix cracking in fibre-reinforced laminates of arbitrary layup are considered in the context of continuum damage mechanics. A complete model for the damage process is accomplished by establishing an appropriate damage representation and a damage growth law. Talreja's damage representation has been modified and significant simplifications have been achieved in defining the damage-related material constants for this particular form of damage in a convenient way. The modified damage representation is lamina-based while Talreja's damage representation is, in the context of this paper, laminate-based. The assumptions introduced to simplify the damage representation are examined and justified. Employing the concept of a damage surface, an incremental damage growth law is formulated. A complete damage model is achieved by combining the damage representation and the damage growth law. The model results in a new laminate theory which describes the deformation of laminates as well as the development of the damage process in the form of crack multiplication. This enables practical predictions to be made of the behaviour of laminated structures made of fibre-reinforced composites experiencing transverse matrix cracking.

Keywords: damage model; transverse matrix cracking; crack multiplication; continuum damage mechanics; fibre-reinforced composites; composite laminates

1. Introduction

In laminated structures composed of uniaxially fibre-reinforced composite laminae, matrix cracking transverse to the plane of the laminae and parallel to the fibres is one of the most common types of damage arising in applications of such structures. Many publications have been devoted to micromechanics-based investigations of this problem (see, for example, Altus & Ishai 1986; Garrett & Bailey 1977; Hashin 1985; Herakovich *et al*. 1988; Highsmith & Reifsnider 1982; Lewinski & Telega 1996*a*, *b*; McCartney 1990; Swanson 1989) although most publications in the literature address only cross-ply laminates. Because of the nature of this type of damage, usually involving a large number of cracks in a characteristic volume of the material called the representative volume, continuum damage mechanics (CDM) has been employed by several authors (Allen *et al*. 1987*a*, *b*; Talreja 1985*a*, *b*, 1986; Talreja *et al*. 1992) to model its effects. A CDM representation of damage expresses mathematically the effects of damage at any given level. Talreja (1985*a*) used a vectorial damage variable, approached the problem of damage in composite materials in a systematic way and applied the results directly to the case of damage resulting from matrix cracks. Applications of the theory can be found in his subsequent publications (Talreja 1985*b*, 1986; Talreja *et al*. 1992). A vectorial damage variable has the advantage of simplicity relative to other damage variables expressed in terms of higher-order tensors but loses generality in reflecting the influence of the spatial geometry of the crack surfaces since it takes account of only the planar projection of the crack surfaces (Krajcinovic 1984). However, for transverse matrix cracks in uniaxially fibre-reinforced composites, the spatial shapes of the crack surfaces do not play a significant role since fairly flat crack surfaces are usually observed. Therefore, Talreja's damage representation using a vectorial damage variable is beneficial for this particular mode of damage.

Talreja's (1985*a*) work appears to be the first paper to apply the theory of CDM systematically to modelling the effects of damage in fibre-reinforced composite materials. As an early attempt at employing CDM to composite materials, Talreja's theory is remarkable. However, in reviewing it, the following comments can be made.

(1) Talreja treated a whole laminate as a single material with which the damage variable is associated. As a result, all the damage-related material constants are defined for this 'material'. Talreja suggested that the damage-related material constants should be determined by experiments. However, these constants depend not only on the properties of the constituent materials in the laminate but also on the layup configuration of the laminate. Therefore, it appears that, before the theory

can be applied to any laminate to describe its behaviour under damage, a series of tests would have to be conducted on a laminate of the same material and the same layup to obtain these constants. This makes the implementation of this theoretical approach impractical because of its heavy dependence on experiments. Consequently, an approach which is essentially lamina-based but incorporates the influence of the presence of adjacent laminae of different fibre orientations is desirable for practical applications.

(2) Another shortcoming of treating a laminate as a material is that it ignores the structural nature of the laminate completely. The position of the cracked lamina in the laminate makes a difference to the behaviour of the laminate. In Talreja's damage representation there is an empirical factor which takes account of the constraints on the crack surface displacements depending mostly on whether the cracked lamina is a surface lamina or is embedded in the laminate. This is obviously insufficient to reflect the complete structural nature of the laminated composite. Although such an approach may be acceptable for cases involving only membrane loading and deformation where the laminate and the damage distribution have middle-plane symmetry, it is not appropriate when, for instance, bending is present. In this case laminae may make significantly different contributions to the overall behaviour of the laminate simply because of their different positions in the laminate. Since one of the motivations of the present work is to implement damage mechanics to general problems in which bending could be one of the major aspects, an approach which takes account of the development of damage in specific laminae and quantifies its effects with due respect to where the cracked lamina is placed in the laminate needs to be sought.

(3) In Talreja's (1985*a*) original work, the damage variable was defined as an unspecified function of crack density (or crack surface area). In his later applications (e.g. Talreja 1985*b*) this function was taken to have a form which results in all the effective material properties being proportional to the crack density. However, from experimental data (Highsmith & Reifsnider 1982) and from the analyses of cracked laminates (Hashin 1985), the effective Young's modulus of the cracked laminate in the direction perpendicular to the cracks showed significant nonlinearity with respect to the crack's density. The same situation arises with the effective in-plane shear modulus of a cracked laminate. Thus, the applicability of Talreja's damage representation is limited to a small range of crack density in which linearity is a reasonable approximation to a nonlinear curve but this range may not be sufficient for practical problems and, therefore, this aspect also needs to be addressed.

In the first part of this paper an attempt is made to examine these practical shortcomings. First, the cracked *lamina* is treated as a material rather than the whole *laminate*. This approach was also employed in Thionnet & Renard (1993), where a meso–macro approach was proposed. The present paper will promote such an approach, to be hereafter referred to as a *lamina-based damage representation*, to a simpler but more developed state. In it, the damage variable is associated with an individual lamina or, in other words, the damage (variable) field is defined in a piecewise manner (from lamina to lamina) over the thickness of the laminate. With the help of this damage representation, the effective material properties of the cracked lamina can be expressed in terms of the damage variable while the overall behaviour of the laminate can then be described by employing a laminate theory. This is seen as the most effective way of minimizing the dependence of the damage representation on experiments. Additionally, it takes account of the influence of the position of the cracked lamina in the laminate within the framework of laminate theory.

As a by-product of this approach, there is no need to introduce so-called multi-vectorial damage variables (Talreja 1985*a*) even though several cracked laminae of different ply-angles are involved in the laminate. As long as the only damage mode is transverse matrix cracking, it can be treated separately at the lamina level while the assembly of all the laminae is governed by a laminate theory.

The nonlinear relation between the effective material properties (in particular, transverse Young's modulus E_2 and in-plane shear modulus G_6) and crack density, which has been ignored in Talreja's later work, can be taken into account. This is achieved by introducing a different measure of damage. There is no reason why the damage measure has to be crack density. If one retains the original damage measure (the length squared of the vectorial damage variable) as in Talreja (1985*a*), the linear relation between the effective material properties and the damage, as embodied in equations (2.1) in the next section, remains unimpaired.

The effective properties of a lamina are affected by the layup of the laminate as a result of the different constraints imposed on the lamina arising from its interaction with surrounding laminae. In any lamina-based approach, there should be a means of allowing this effect to be taken into account. This is one of the important aspects of the proposed damage representation.

Having established a damage representation, a damage model for transverse matrix cracking can be completed by introducing a damage growth law describing the evolution of damage. The concept of a damage surface has been introduced for establishing damage growth laws in several studies (e.g. Krajcinovic & Fonseka 1981). This plays a similar role to the yield surface in the theory of plasticity. However, as was noted in Krajcinovic & Fonseka (1981), the construction of damage surfaces has suffered from the lack of experimental data, and the process has had to be based on a number of arbitrary assumptions. Removing these arbitrary assumptions from the theory is not an easy task, and is unlikely to be achieved for general cases in the short term. It therefore seems appropriate at this stage to establish particular, instead of general, damage growth laws for specific problems using the concept of the damage surface.

A damage process involves two aspects, the initiation and the growth of damage. The latter includes both the development in severity and the expansion of the damaged zone. Consider damage-zone expansion first. This is a typical structural feature resulting from non-uniform stress distributions and the transition from an undamaged state to a damaged state at a material point as a result of load shedding and local increases in stress. It is essentially a damage initiation process as far as this material point is concerned and will be treated as such herein. A material 'point' is synonymous with a representative volume, which is a compromise between a volume of material small enough for the non-uniformity in stress distribution to be neglected (infinitesimal macroscopically), and large enough for the discrete features of the damage to be smeared within it (infinitely large microscopically). With the help of this concept, one can concentrate on establishing a constitutive relationship governing the damage process in a representative volume and leave the structural behaviour, i.e. the expansion of the damaged zone and the spatial non-uniformity of deformation, to a structural analysis using, for example, finite elements. Thus within a typical representative volume, the damage process has two elements, its initiation and its development in severity.

Confining attention to the problem of transverse matrix cracking in a representative volume, damage development in severity takes the form of crack multiplication. The assumption of the existence of a damage surface unifies the initiation and the development in severity. The damage process is determined completely by the internal state in the material of the representative volume and the history of achieving this state. The internal state is described by stresses, strains and the damage. A damage surface is a function defined in the internal state variable space. One might suggest from the perspective of physical cracking processes that a crack at a point can result from one of the two different processes: the initiation of a new crack; and elongation of an existing crack from a neighbouring representative volume. They may appear to be two completely different mechanisms. The latter can be described as crack propagation using the terminology of fracture mechanics. Normally crack propagation is largely dependent on the crack length but, in composite laminates, there is evidence that this crack-length dependence is eliminated by the constraints provided by the laminae neighbouring the cracked lamina (Ogin & Smith 1987). This is one of the most important reasons for using laminated materials. It is, therefore, a natural inference that the extension of existing cracks into uncracked regions is more to do with the local stress state than how far the crack has grown from its original point of initiation. In this sense, crack initiation and crack extension from pre-existing cracks are both determined by the local stress state. One could still argue that the two phenomena correspond to different levels of the internal state. This is a matter to be considered in the future development of the model. It is assumed here, for the sake of simplicity, that they both correspond to the same internal state. Thus, the onset of damage in the material at a point is predicted by the stress state at the point which can be obtained from a laminate analysis without involving any other considerations.

Applying the above argument to the initiation of cracking at a point, the initial failure in conventional laminate analysis, often referred to as first ply failure, can be employed. A more sophisticated approach would take account of the dependence of the *in situ* strength of the ply on the thickness of the ply (Fan & Zhang 1993). While there is no major obstruction to doing this, the effects will be ignored in this paper for the sake of simplicity.

The growth of damage in the form of crack multiplication in a representative volume is the main feature of damage growth which is considered herein. The argument made above is to justify that the cracks in this representative volume can be considered as full length (in the plane of the lamina) cracks. The damage process is then idealized by the problem of the generation of a third crack between two existing neighbouring cracks. Further cracking like this is controlled by the stress state in the material between the two existing cracks, although stress analyses of cracked laminates (Hashin 1985; Li *et al*. 1994) show that the highest stress intensities always appear in local regions around the crack tips at interlaminar interfaces. The high stresses around the crack tips are relevant to the whole damage process, especially to the initiation of other damage modes (Altus & Ishai 1986), but are unlikely to affect the crack multiplication process, and, therefore, their effects will be ignored here. The region of second-highest stress level is in the middle between two cracks and, therefore, a new crack is most likely to appear there. This implies that the increase in crack density is by a doubling process rather than by continuous growth. However, in reality, the process of crack multiplication appears to be defect-sensitive and more often than not cracks are initiated from voids. Due to the random distribution of defects in the material, the cracks will not be regularly spaced even if the nominal global stresses are uniform. The cracks which are generated between existing pairs of cracks in a uniformly loaded lamina will, therefore, not appear at the same time but will do so separately, so that a continuously varying crack density increase can be justified in a statistical sense, reflected by an increase in the damage variable. Within the continuum idealization, the crack density, and hence the damage variable, is assumed to be a field variable defined at every point in the material indicating, in a statistical sense, the distribution of cracks in the representative volume surrounding the point.

2. A lamina-based damage representation for damage due to transverse matrix cracks

Given a laminate containing transversely cracked laminae, the major assumption introduced in the present damage representation is that the behaviour of all the laminae, cracked and uncracked, can be effectively described by plane-stress states. The implication is that the effects of the out-of-plane stresses including the transverse (through-thickness) direct stress and the two transverse shear stresses can be neglected. These stresses are present in reality as a result of the presence of the transverse matrix cracks, especially around the crack tips, and, therefore, the assumption needs to be justified. This will be dealt with after the outcome of the assumption has been displayed.

As indicated in § 1, a lamina-based damage representation is defined for each individual lamina. An isolated lamina breaks at its weakest position, usually associated with defects in the material before extensive cracks develop when it is subjected to stress transverse to the fibres. The development of extensive transverse matrix cracks in a lamina, therefore, takes place only when the lamina is contained within a laminate. In dealing with an individual lamina here, in order to introduce the lamina-based damage representation, the lamina should be understood to be a portion isolated from the laminate in the sense of defining a classical free-body in statics. However, the plane-stress assumption just introduced implies that the lamina is only subjected to in-plane stresses but is capable of sustaining multiple cracks. Treating such an idealized cracked lamina as a material, its constitutive relations can be established with the help of the Helmholtz free energy, a state function of the state variables (the strain tensor components and the damage vector) subject to the constraints imposed by the second law of thermodynamics (Coleman & Noll 1963). For problems involving small strain and small damage (the scale depends on the nature of the problem and smallness does not necessarily mean a value far smaller than unity), the Helmholtz free energy can be approximately expressed in the form of a Taylor's expansion truncated at the second-order terms of strains and the damage. Use can be made of the integrity basis of invariants of the internal state variables (the strain tensor and the damage vector) to minimize the number of material constants by taking account of all the symmetries the material exhibits (Pipkin & Rivlin 1959). This procedure has been performed by Talreja (1985*a*) and, for the material of a lamina containing transverse cracks in a plane-stress state, it results in effective material properties expressed as

$$\left.\begin{aligned} E_1 &= E_1^0 + a_1\omega, \qquad \nu_{12} = \nu_{12}^0 + a_4\omega, \\ E_2 &= E_2^0 + a_2\omega, \qquad \nu_{21} = \nu_{21}^0 + a_5\omega, \\ G_6 &= G_6^0 + a_3\omega, \end{aligned}\right\} \tag{2.1}$$

where E_1, E_2, ν_{12} and ν_{21} are the longitudinal and transverse Young's moduli and Poisson's ratios, respectively, and G_6 is the in-plane shear modulus, in a damaged state. A superscript 0 refers to values in the undamaged state. ω is hereafter termed the *damage parameter* and can be regarded as the length (squared) of the vectorial damage variable. The constants a_i (i = 1–5) are introduced as a set of damage-related material constants. In Talreja's (1985*a*) original expressions, the above effective material properties are associated with another set of damage-related constants. Those constants are coefficients in the expression for the Helmholtz free energy and have no direct physical meaning in terms of identifying them in mechanical tests, while the a_i in equations (2.1) can be easily deduced in principle since they are the rates of change of these material properties with respect to the damage parameter. One can relate constants a_i in equations (2.1) to those introduced by Talreja through algebraic linear transformations.

It is relevant to note that the *damage parameter* is a scalar but it should be distinguished from the type of *scalar damage variables* referred to in some damage representations (Krajcinovic 1984). The scalar characterization in terms of the damage parameter ω, becomes possible as a result of the particular damage mode, transverse matrix cracking, because the damage vector associated with it has a fixed orientation and the only thing which varies is the length of the vector.

Ideally, the damage-related material constants in equations (2.1) would be determined by experiments on a damaged lamina. However, as has been pointed out earlier, it is not possible to obtain a single lamina cracked at various levels experimentally. A controlled damaged state can only be achieved when the lamina is embedded into a laminate with other laminae of different fibre orientations or with different materials. It would, therefore, appear necessary to establish a new testing procedure involving laminated specimens from which information about the damaged lamina could be extracted. Such a procedure is not available yet and alternative ways of determining these constants are examined in this paper.

Implicit in equations (2.1) is the orthotropy of the cracked lamina. This is a natural inference from the plane-stress state assumption. The material's topological symmetries are not altered by the presence of cracks transverse to the lamina and parallel to the fibres. Conversely, if the effective properties of a cracked lamina in a laminate can be obtained, the orthotropy in the effective material properties provides one of the checks on the plane-stress state assumption. This will be pursued in the next section.

If the temperature is included in the Helmholtz free energy as an independent state variable, the damage representation for the thermal expansion coefficients can be obtained correspondingly. This could be incorporated in the model in a straightforward manner but, given the complexity of the model as presented herein, this will be left for a future publication.

One important feature which will not be included is the bi-modular behaviour resulting from the crack-closing effect when the lamina is subjected to compression perpendicular to the cracks. Taking this into consideration, the expressions for the damaged material properties would have to be altered by defining them in a piece-wise manner. This has not been investigated in any previously published work and is seen as a future development. The basic principles would be the same as in the cases treated herein.

(*a*) *The damage parameter*

The damage parameter, the length squared of the damage vector in Talreja's (1985*a*) work, is defined in general as a function of the crack surface area in a representative volume of damage material. Transverse matrix cracks in a uniaxially fibre-reinforced lamina embedded in a laminate, usually span the full thickness of the lamina. This observation implies that the crack surface area is proportional to the crack density in the representative volume. Thus the damage parameter can be considered as a function of crack density. Talreja (1985*b*) chose a simple form for the function, namely a damage parameter proportional to the crack density. The price for this simplicity is the introduction of an extra restriction on the smallness of the damage in addition to those introduced when the higher-order terms in the expansion of the Helmholtz free energy are neglected. This may narrow the range of applicability of the damage representation.

Talreja's simple form for the damage parameter will be discarded here. Instead, the damage parameter will be associated directly with the relative change in the transverse Young's modulus of the lamina. In equations (2.1), a different definition of ω may result in different values for the damage-related material constants, a_i, while maintaining fixed ratios between these constants. In other words, the damage-related constants, a_i, are determined to within a constant factor. Absorbing an appropriate common factor into ω, the damage parameter can be defined as the relative change in the transverse Young's modulus,

$$\omega = 1 - E_2/E_2^0. \tag{2.2}$$

With this damage parameter, the effective material properties remain as linear functions of the damage parameter. The nonlinear relation between the effective material properties and the crack density, as shown in Highsmith & Reifsnider (1982) and Hashin (1985), is taken into account by the relation between the damage parameter and the crack density. It is expected that the present damage representation can be applied to a wider range of damage levels than that described by Talreja while remaining in the small damage regime imposed by retaining only terms up to the second order in the Helmholtz free energy.

A shortcoming of the proposed damage parameter is that it is less observable than the crack density in an experiment and, since it depends on the layup of the laminate, different values may be obtained in different laminates even though the lamina is cracked to the same crack density. However, with the help of a method for analysing a cracked laminate (see, for example, Li *et al*. 1994), correlation between the damage parameter and the crack density can be established. It can be further simplified if an appropriate curve-fit of this relation is introduced to express the ω versus crack density relation in an analytical manner for the particular laminate (Thionnet & Renard 1993).

With the damage parameter so defined, we can proceed to determine the damage-related material constants a_i. From the nature of their dependence on the laminate layup, they can be classified into two groups, one group is *layup-independent*, i.e. completely determined by the material properties of the cracked lamina itself, and the other is *layup-dependent*, i.e. influenced by neighbouring laminae.

(*b*) *Layup-independent constants*

When the damage parameter is defined by equation (2.2), the constant a_2 in equations (2.1) can be determined immediately and is simply

$$a_2 = -E_2^0. \tag{2.3}$$

The effective transverse Young's modulus of a cracked lamina, E_2, is obviously affected by where the lamina is embedded into the laminate. The layup-dependent nature of E_2 is taken into account through the damage parameter but the constant a_2 is clearly a layup-independent, damage-related (but independent of damage) material constant.

Consider a free-body lamina with transverse matrix cracks in it. Since it is assumed to be in a plane-stress state, the constants a_1, a_4 and a_5 can be easily determined. The Young's modulus and the Poisson's ratio along the fibres are not expected to be affected by the existence of cracks (assuming ideal cracks) parallel to the fibres and, therefore, remain constant. Thus,

$$a_1 = 0 \tag{2.4}$$

and

$$a_4 = 0. \tag{2.5}$$

The constant a_5 is not an independent constant and it can be expressed in terms of a_1, a_2 and a_4 from the reciprocal relation

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}, \tag{2.6}$$

resulting in

$$a_5 = -\nu_{21}^0. \tag{2.7}$$

A basic assumption in CDM is that a damaged material at a fixed level of damage can be treated as an ordinary material with certain effective material properties. Because of this, equation (2.6) must be satisfied as the condition for the existence of the strain-energy density function of the fictitious material (Sokolnikoff 1956).

The damage-related constants determined in equations (2.3), (2.4), (2.5) and (2.7) are either zero or related to some conventional material properties (in the undamaged state) of the lamina concerned. They are obviously independent of the laminate layup and, therefore, are termed layup-independent constants.

(*c*) *Layup-dependent constants*

Consider the last constant a_3 in equations (2.1) which is associated with shear. Re-writing the third expression of equations (2.1) as

$$G_6 = G_6^0(1 - k\omega), \tag{2.8}$$

one obtains

$$a_3 = -kG_{12}^0. \tag{2.9}$$

This expresses the constant a_3 in terms of k, the ratio of the relative changes in the in-plane shear modulus and the transverse Young's modulus, as a result of damage

$$k = \left(\frac{G_6 - G_6^0}{G_6^0}\right) \Bigg/ \left(\frac{E_2 - E_2^0}{E_2^0}\right). \tag{2.10}$$

There is no *a priori* reason to assume k to be layup independent. It is, therefore, treated as a layup-dependent constant which depends on the material as well as the layup of the laminate. The constant k is the only constant which needs to be determined (ideally by experiment) for implementation of the damage representation in cases where shear is involved. Unfortunately, no experimental data related to the shear behaviour of cracked laminates are available in the literature to the authors' knowledge. Indeed, given the layup-dependent nature of this constant, the practical difficulties in determining it for all circumstances are formidable. As an alternative, in the next section a theoretical approach for calculating k is described.

Obviously, equation (2.8) is invalid when the damage becomes large. As an extreme case, when cracks are infinitely dense, the rigidity of the cracked lamina perpendicular to the cracks vanishes and, hence, $\omega = 1$. The in-plane shear modulus must vanish correspondingly. This requires that $k = 1$. However, the values of k obtained from cases corresponding to smaller values of ω are substantially different from unity. This suggests a nonlinear dependence of the effective shear modulus on the damage parameter for large damage, and k can be approximated as a constant only when the damage is small. To take account of the nonlinearity at large damage, higher-order terms would have to be included in the Helmholtz free energy. This is beyond any established development of CDM and will not be pursued in the present paper.

As a result of the simple argument above, the damage representation for transverse matrix cracks in a lamina of a laminate can be summarized as

$$E_1 = E_1^0, \quad E_2 = E_2^0(1-\omega), \quad G_6 = G_6^0(1-k\omega), \quad \nu_{12} = \nu_{12}^0 \quad \nu_{21} = \nu_{21}^0(1-\omega). \tag{2.11}$$

3. Effective material properties and verification of the damage representation

This section is devoted to the verification of the proposed damage representation, and the justification of the plane stress state assumption which has been introduced. This is achieved by obtaining the effective material properties of the laminae, both cracked and uncracked, involved in a laminate and comparing them with those obtained following the proposed damage representation, equations (2.11). An approach which is independent of the damage representation will be used for this purpose. This is a micromechanical analysis of a cracked laminate using the finite strip method (Li *et al*. 1994) in which the detailed stress and strain distributions in cracked laminates of any layup sequence and subjected to general loading (any combination of membrane and bending loads) can be calculated. The method analyses a typical segment of the laminate bounded by two neighbouring cracks in the lamina of interest. The characterization of the damage in the form of transverse matrix cracking involves two steps. The first is a fictitious rearrangement of cracks distributed at random into a regularly spaced state from which the representative volume is taken. Though not reflecting reality, it is a commonly accepted idealization (Altus & Ishai 1986; Garrett & Bailey 1977; Hashin 1985; Herakovich *et al*. 1988; Highsmith & Reifsnider 1982; Li *et al*.

1994; McCartney 1990; Swanson 1989; Thionnet & Renard 1993) and will be adopted here. The second step is to smear out the discrete cracks into a degraded continuum with material properties determined by a damage representation. This can be justified by comparisons between the effective properties of the laminae obtained from a micromechanical cracked laminate analysis which takes account of the interlaminar interactions (including the high stress concentration around the local regions of crack tips) and those proposed in the damage representation. In other words, the former serves as a numerical experiment.

From the micromechanical cracked laminate analysis (Li *et al.* 1994), the effective properties of a lamina can be obtained from an energy equivalence argument. From the strain distributions in each lamina, the energies stored in the lamina corresponding to given deformations can be calculated. These can be used to extract the effective material properties of the lamina. When the following six sets of the generalized membrane strains, ϵ_x, ϵ_y and ϵ_{xy},

$$\left.\begin{aligned} \epsilon_x &= 1, & \epsilon_y = \epsilon_{xy} &= 0,\\ \epsilon_y &= 1, & \epsilon_x = \epsilon_{xy} &= 0,\\ \epsilon_{xy} &= 1, & \epsilon_x = \epsilon_y &= 0,\\ \epsilon_x = \epsilon_y &= 1, & \epsilon_{xy} &= 0,\\ \epsilon_x = \epsilon_{xy} &= 1, & \epsilon_y &= 0,\\ \epsilon_y = \epsilon_{xy} &= 1, & \epsilon_x &= 0, \end{aligned}\right\} \tag{3.1}$$

are applied to the laminate, the respective energies stored in each lamina, U_1, U_2, U_6, U_{12}, U_{16} and U_{26} can each be calculated.

On an energy-equivalent basis, the elements of the effective laminar stiffness matrix $[q]$ in the laminate axes can be obtained from

$$\left.\begin{aligned} q_{11} &= 2U_1/V,\\ q_{22} &= 2U_2/V,\\ q_{66} &= 2U_6/V,\\ q_{12} &= (U_{12} - U_1 - U_2)/V,\\ q_{16} &= (U_{16} - U_1 - U_6)/V,\\ q_{26} &= (U_{26} - U_2 - U_6)/V, \end{aligned}\right\} \tag{3.2}$$

where V is the magnitude of the representative volume. This stiffness matrix $[q]$ is given in laminate axes (aligned with the loading direction usually) and can be transformed to that, $[Q]$, in the material axes (the elastic principal axes of the lamina (Tsai & Hahn 1980)) by

$$Q = T^{-1} q T^{-\mathrm{T}}, \tag{3.3}$$

where T is the coordinate transformation matrix,

$$T = T(\theta) = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & 2\cos\theta\sin\theta \\ \cos\theta\sin\theta & -\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}. \tag{3.4}$$

$T^{-1} = T(-\theta)$ and θ is the orientation of the material axes relative to the laminate axes. The effective material properties, such as Young's moduli, shear modulus

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Table 1. *Material properties and layup definition of the laminates involved*

	laminate 1 glass/epoxy (Highsmith & Reifsnider 1982)	laminate 2 AS4/3501-6 (Talreja *et al.* 1992)	laminate 3 Tactix 556 (Talreja *et al.* 1992)	laminate 4 Tactix 695 (Talreja *et al.* 1992)	laminate 5 glass/epoxy (Li *et al.* 1993)
layup	$[0°/90°_3]_s$	$[0°_2/90°_2]_s$	$[0°_2/90°_2]_s$	$[0°_2/90°_2]_s$	[55°/−55°/55°/−55°]
t	0.203 mm	0.149 mm	0.140 mm	0.149 mm	0.25 mm
E_1	41.7 GPa	140.1 GPa	151.1 GPa	136.8 GPa	45.60 MPa
E_2	13.0 GPa	8.36 GPa	7.09 GPa	6.93 GPa	16.23 MPa
ν_{12}	0.30	0.253	0.241	0.268	0.278
G_4	4.58 GPa	3.20 GPa	2.72 GPa	2.57 GPa	5.50 MPa[a]
G_6	3.40 GPa	4.31 GPa	3.63 GPa	3.30 GPa	5.50 MPa
tensile strength along fibre	1170 MPa[b]	—	—	—	1280 MPa
tensile strength transverse to fibre	32 MPa[b]	—	—	—	40 MPa
shear strength	45 MPa[b]	—	—	—	73 MPa

[a]Assumed value; [b]interpolated values; —, values not available.

Table 2. *Effective Q matrix* (GPa)

crack density, δ	Q_{11}	Q_{22}	Q_{12}	Q_{66}	Q_{16}	Q_{26}
0.00	42.90	13.38	4.01	3.40	0.00	0.00
0.25	42.40	7.76	2.33	2.71	−0.05	−0.16
1.00	41.90	2.27	0.68	1.39	−0.02	−0.07

and Poisson's ratios, can then be extracted from $[Q]$. They will serve as data from numerical experiments since the way they are calculated is independent of the damage representation. The damage representation will then be validated against these numerical experiments in the next section. The first step is to verify the orthotropy of a damaged lamina which is implied in equations (2.1). Effective material properties of damaged laminae will be calculated for a series of given crack densities, which can be used to verify the values of a_i determined intuitively in the previous section.

(*a*) *The orthotropy of a cracked lamina in a laminate of arbitrary layup*

A lamina is orthotropic if the elements Q_{16} and Q_{26} of $[Q]$ are zero or negligible compared to other elements. This is equivalent to the statement that the material possesses reflectional symmetries about planes perpendicular, respectively, to axis-1 (along the fibres) and axis-2 (transverse to the fibres and in the plane of the lamina). A single lamina containing cracks transverse to the lamina and parallel to the fibres is obviously orthotropic as far as its effective properties are concerned. Furthermore, a cracked lamina embedded in a cross-ply laminate is bound to be orthotropic because these symmetries exist for both the cracked lamina and its surrounding laminae and should the interlaminar interactions be significant they would not affect these topological symmetries. What needs to be examined is the case when a cracked lamina is embedded in a laminate having off-axis laminae because the local stress distributions in the vicinity of the cracked lamina do not show the required symmetries. As an example, consider a symmetrical glass/epoxy laminate, $[30°/90°]_s$, and suppose that the two internal 90° laminae are cracked. When the laminate is subjected to uniaxial strain, for instance, ϵ_x, which is reflectionally symmetric about the crack's surfaces, non-reflectional shear stresses in these 90° laminae develop within the local regions around the cracks along the interfaces with the 30° laminae. If their effects became significant, they could undermine the orthotropy of the effective material properties. It will be shown that this does not happen. Take the material properties for the postulated laminate (non-cross-ply) from those of the constituent laminae in laminate 1 in table 1. The elements of $[Q]$ deduced from equations (3.2) and (3.3) by the energy equivalent technique are listed in table 2 for several typical crack densities. It is seen that Q_{16} and Q_{26} are indeed negligible (the values may even be numerical rounding errors). Other examples show similar trends. This indicates that the crack-induced out-of-plane stresses only have localized effects, which disperse the stress concentrations caused by material discontinuities, but the orthotropy of the effective properties of the lamina, which represents one of its main global characteristics, is not affected significantly. Therefore, if the cracked lamina is treated as a material, it can be considered to be orthotropic insofar as its effective material properties are concerned.

Because the cracked lamina is, therefore, effectively orthotropic, its effective characteristic properties can be extracted from $[Q]$ by inverting the stiffness-material property relation,

$$\left.\begin{aligned} E_1 &= (Q_{11}Q_{22} - Q_{12}^2)/Q_{22}, \\ E_2 &= (Q_{11}Q_{22} - Q_{12}^2)/Q_{11}, \\ G_6 &= Q_{66}, \\ \nu_{12} &= Q_{12}/Q_{22}. \end{aligned}\right\} \tag{3.5}$$

The significance of the result obtained above is that laminates made up of off-axis laminae, i.e. non-cross-ply laminates, do not present any difficulty to the proposed damage representation because the damage in a lamina does not alter its material orthotropy with respect to its principal axes. A lamina-based damage representation can take advantage of this result by employing a laminate theory to describe the behaviour of any assembly of laminae, damaged or not, irrespective of layup. Notwithstanding this important result, cross-ply laminates will be considered mainly because of the availability of experimental results for them in the literature, while the model's applicability to non-cross-ply laminates will be illustrated through an example in § 7. Four laminates will be cited below, one tested by Highsmith & Reifsnider (1982) and analysed by Hashin (1985) and three by Talreja *et al*. (1992). The material properties of each of these laminates are listed in table 1 where the layup and the ply-thickness, t, involved are also given.

(*b*) *The relationship between the damage parameter and the crack density*

The relationship between the damage parameter $\omega = 1 - E_2/E_2^0$ of a cracked lamina (full thickness of all the adjacent 90° plies) and the crack density δ (non-dimensionalized with respect to the thickness of the cracked lamina, e.g. $6t$ for lami-

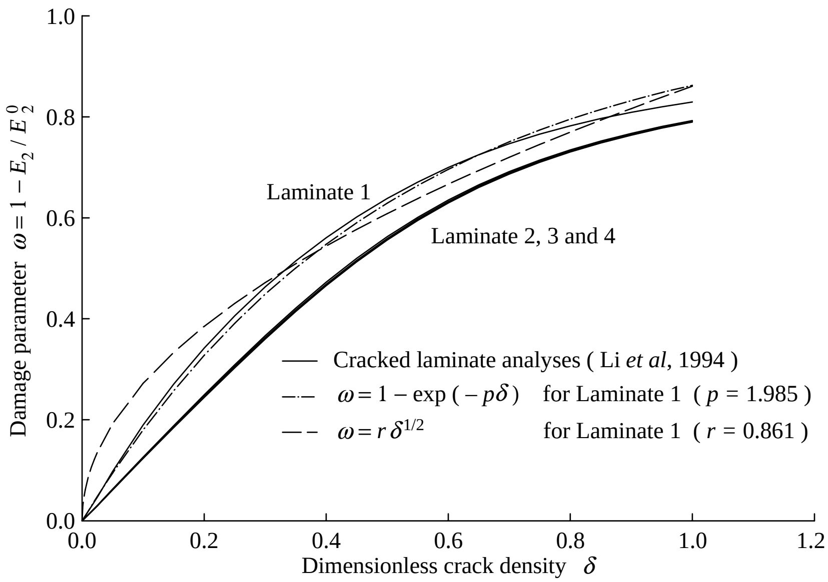

Figure 1. Damage parameter versus dimensionless crack density.

nate 1 and $4t$ for laminates 2, 3 and 4) can be established using the micromechanical cracked laminate analysis (Li *et al*. 1994). The procedure is to find the effective property E_2, following equations (3.2), (3.3) and (3.5), at a given crack density δ, and to calculate ω according to equation (2.2). Figure 1 shows the calculated ω versus δ curves for the four cracked laminates. The three curves corresponding to the laminates tested in Talreja *et al*. (1992) (laminates 2, 3 and 4 in table 1) are almost coincident. Nonlinearity is apparent between the two characteristic parameters ω and δ. While a linear relationship may be suitable for low crack densities, say below $\delta = 0.5$, the nonlinearity becomes non-negligible at higher crack densities. The direct use of crack density as the damage measure (Talreja 1985*b*) means that the damage representation is unable to reflect reasonably the change of the effective Young's modulus of the laminate beyond the range of the almost-linear part of the corresponding curves, even if the damage is still within the small damage range imposed by ignoring higher-order terms in the Helmholtz free energy expression. This confirms that using the crack density as a direct measure of damage in the damage representation puts an extra limitation on the applicability of the damage representation. Using the damage parameter defined in equation (2.2) removes this limitation.

Thionnet & Renard (1993) suggested a simple empirical relation between ω and δ,

$$\omega = r\delta^{1/2}, \tag{3.6}$$

where r is a constant obtainable by curve fitting using the data from the micromechanical cracked laminate analysis. For laminate 1 the fitted curve to equation (3.6) is shown in figure 1 by the dashed line and r is found to be 0.861.

It has also been shown in figure 1 that a function of the form,

$$\omega = 1 - \exp(-p\delta), \tag{3.7}$$

gives a better fit for this particular case, plotted by the dash–dotted line, where p is a constant to be determined by curve fitting and equal to 1.985 in this particular

Table 3. *The damage parameter ω in the cracked lamina*
(Parametric study of the effects of the uncracked laminae.)

amount of parametric change	parameters subject to parametric change: E_1 (uncracked laminae)	E_2 (uncracked laminae)	ply thickness (uncracked laminae)
double	0.602	0.624	0.618
original	0.628	0.628	0.628
half	0.654	0.630	0.635

case. Equation (3.7) improves the fit not only in terms of the standard deviation but also by removing the slope singularity at $\delta = 0$ in relation (3.6).

Curve fitting, such as that described above, can be used to provide a simple analytical relation between ω and δ. This, however, does not constitute any part of the damage model. The property it conveys is that there exists a one-to-one relation between ω and δ, curve fitting being a simple approximation to the form of that relationship. More accurate forms can be obtained numerically as long as the relationship exists. Crack densities are particularly useful in comparing theoretical predictions with experimental data.

An important result from Thionnet & Renard (1993) is that the constant r, or to be precise, the damage parameter ω, is intrinsic to the material of the lamina itself and not to the other undamaged laminae in the laminate. Parametric studies have been carried out by the present authors on a $[0°/90°]_s$ laminate with the same material properties as laminate 1 in table 1, in which the 90°-ply is assumed to be cracked. At a fixed crack density, $\delta = 0.5$, parameters such as the thicknesses and the Young's moduli E_1 and E_2 of the 0°-ply were changed to both double and half of their original values, respectively. The results are given in table 3 and show negligible changes (below 5%) in ω of the cracked lamina. The hierarchy of sensitivity to these parameters in the uncracked lamina (0°-ply) in descending order is E_1, thickness and then E_2. If these minor changes as such are neglected, the damage parameter ω conforms to the intrinsic nature of the damage parameter as proposed in Thionnet & Renard (1993). Thus, the layup-dependent nature of ω is simply a matter of whether the cracked lamina lies on the surface of the laminate or is embedded inside of it. In other words, the damage parameter is uniquely defined, i.e. for a given cracked lamina with a given crack density, the value of ω in it will hardly be influenced by the existence and the development of the damage in any other laminae. However, this does not exclude the interaction between the laminae in terms of damage growth. The existence of damage in one lamina does affect the growth of the damage in another lamina, as will be shown in § 6.

(*c*) *Effective along-fibre Young's modulus and Poisson's ratio of a cracked lamina*

From the cracked laminate analysis (Li *et al*. 1994) of the four laminates, it can be shown that the effective Young's modulus E_1 and the effective Poisson's ratios ν_{12} along the fibres (parallel to the crack surfaces) of the cracked lamina in each of the four laminates analysed are not affected at all by the presence of transverse matrix

Table 4. (*a*) *Young's modulus parallel to cracks in cracked laminae* (GPa)

crack density, δ	laminate 1	laminate 2	laminate 3	laminate 4
0.00	41.7	140.1	151.1	136.8
0.25	41.7	140.1	151.1	136.8
1.00	41.7	140.1	151.1	136.8

(*b*) *Poisson's ratio parallel to cracks in cracked laminae*

crack density, δ	laminate 1	laminate 2	laminate 3	laminate 4
0.00	0.300	0.253	0.241	0.268
0.25	0.300	0.253	0.241	0.268
1.00	0.300	0.253	0.241	0.268

cracks and the change in crack density. Table 4 lists these effective properties of the cracked laminae at some typical crack densities and demonstrates their constancy.

Although this exercise has been carried out for the laminates listed which are all cross-ply, the results here will be generally true for laminates other than cross-ply. The only different aspect a non-cross-ply layup presents is the possibility of violation of the orthotropy of the effective material properties of its constituent laminae as a result of the asymmetric stresses induced by the off-axis plies. However, this has been addressed in § 3 *a* where it has been shown that the orthotropy is not affected. Therefore, considering only cross-ply laminates here does not put any restriction on applying the results to laminates of other layups.

Thus as far as the two effective properties, E_1 and ν_{12}, are concerned, the interactions between a cracked lamina and its surrounding are not likely to affect them. From this it can be seen that the values of the constants a_1 and a_4 for a cracked lamina in a laminate are indeed negligible as proposed in the damage representation in the previous section. This also confirms that the constants a_1 and a_4 are indeed layup-independent.

(*d*) *The effective along-fibre shear modulus of a cracked lamina*

The effective in-plane shear modulus G_6 of cracked laminae can also be obtained from the cracked laminate analysis (Li *et al*. 1994). For the same laminates as above, the results shown in figure 2*a* indicate a nonlinear relation between the relative change of the in-plane shear moduli and the crack density of a similar form to that for the transverse Young's modulus E_2 shown in figure 1. However, when the relative changes of the in-plane shear modulus G_6 of the cracked laminae are plotted with respect to the relative change of transverse Young's modulus E_2, i.e. the damage parameter ω, as shown in figure 2*b*, a reasonably linear relationship can be seen. If these curves are replaced by properly fitted straight lines, the errors introduced will be very small as long as ω is, say, less than 0.7. A linear relation between the relative change in the in-plane shear modulus and the relative change in the Young's modulus transverse to the cracks conforms to equation (2.8).

Having established this linearity there is no implication that the effective in-plane shear modulus, G_6, of a cracked lamina is not influenced by its surrounding. In fact, the layup-dependent nature of G_6 is undeniable and the channel through which the

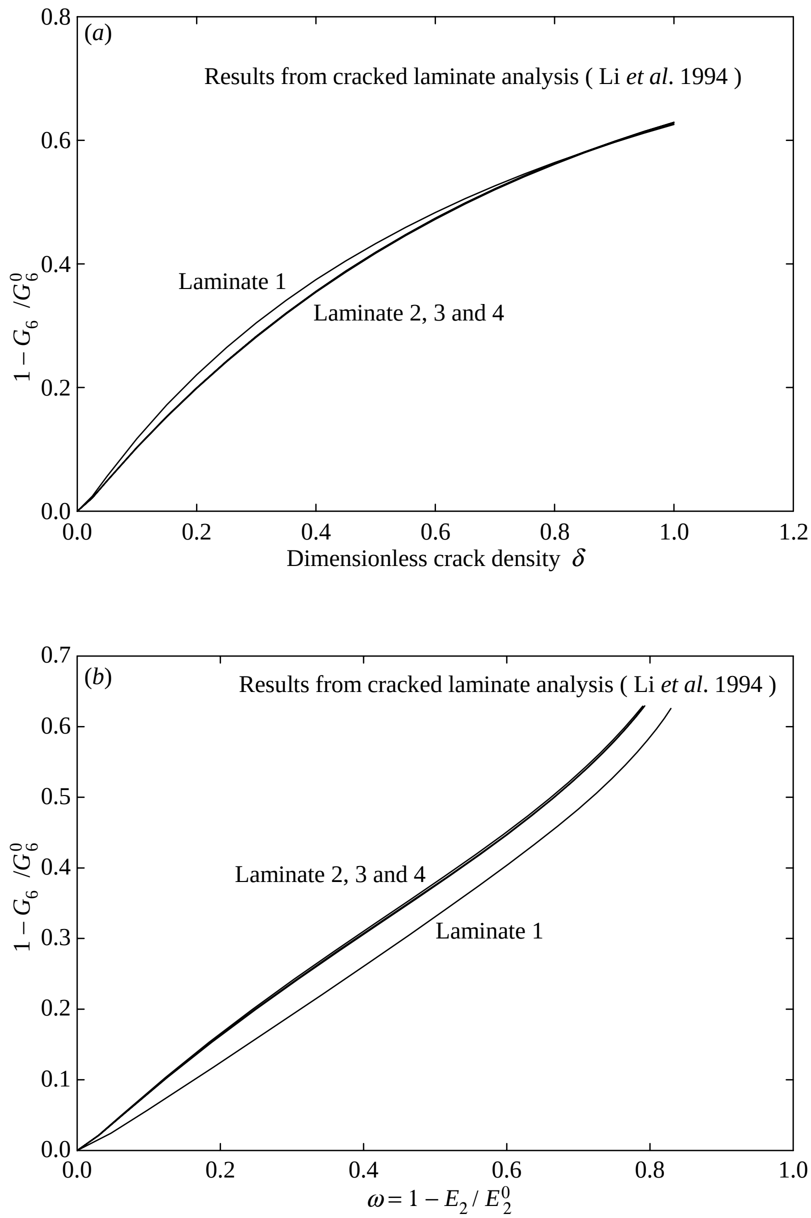

Figure 2. (*a*) Change of the relative shear modulus of the cracked lamina versus dimensionless crack density. (*b*) Change of the relative shear modulus of the cracked lamina versus damage parameter.

influence of the surroundings is imposed is the proportionality factor k. Different layups, and hence different constraints on the cracked lamina from its surrounding, may result in different values of k. After a linear fit has been made to the data obtained from the cracked laminate analysis, the constant k can be determined. Table 5 presents two values of k fitting the data resulting from micromechanical

Table 5. *Parameter k*

	laminate 1	laminate 2	laminate 3	laminate 4
$\delta < 0.5$	0.678	0.751	0.761	0.754
$\delta < 1.0$	0.744	0.765	0.767	0.765

Table 6. *Parameter k in the cracked lamina*
(Parametric study of the effects of the uncracked laminae.)

	parameters subject to parametric change	
amount of parametric change	G_6 (uncracked laminae)	ply thickness (uncracked laminae)
double	0.588	0.636
original	0.636	0.636
half	0.691	0.657

cracked laminate analysis for each of the four laminates up to two different values of δ, namely 0.5 and 1.0. If the linearity were perfect, there would be no difference between these two values from curve-fitting over the different ranges. The difference between them indicates the severity of the nonlinearity in G_6–ω relation which, even so, does not seem to be excessive within the given ranges.

Similar parametric studies to those referred to in § 3 *b* show that changes in k may rise 10% or higher when changes in the thickness of the uncracked lamina or the shear modulus of double and half of their original values, respectively, are made, as indicated in table 6. It seems that the effective shear modulus of the cracked lamina is more sensitive to the changes in the surrounding laminae than the effective transverse Young's modulus, although it is still reasonable to neglect them to a first approximation. Such an approximation reduces the layup-dependent nature of k to the same level as ω, it being a matter of whether the cracked lamina is a surface lamina or an embedded one. To determine this constant, again use can be made of a cracked laminate analysis such as the one in Li *et al.* (1994) and it can be determined from a simple straight-line fit to the data obtained from such analyses for a series of given crack densities. Obviously, this can be performed independently of the analysis of the cracking process but it has to be completed before the cracking process analysis since this constant is required as input data to such an analysis.

It should be noted that the possibility of determining the constant k computationally does not undermine the importance of experimental determination of this constant. On the contrary, experimental data are urgently required for checking these theoretical predictions.

(*e*) *The effective material properties of uncracked laminae in a cracked laminate*

The purpose behind the damage representation proposed is to replace the cracked lamina in a laminate by a fictitious material whose properties are defined by the damage representation so that conventional laminate theory can be used to describe the overall behaviour of the cracked laminate. It has already been shown that the

Table 7. (*a*) *Young's modulus parallel to fibres in the uncracked lamina* (GPa)

crack density, δ	laminate 1	laminate 2	laminate 3	laminate 4
0.00	41.7	140.1	151.1	136.8
0.25	43.9	140.3	151.3	137.0
1.00	42.6	140.3	151.3	137.0

(*b*) *Young's modulus perpendicular to fibres in uncracked laminae* (GPa)

crack density, δ	laminate 1	laminate 2	laminate 3	laminate 4
0.00	13.00	8.36	7.09	6.93
0.25	13.11	8.37	7.10	6.94
1.00	13.04	8.37	7.10	6.94

(*c*) *Shear modulus in uncracked laminae* (GPa)

crack density, δ	laminate 1	laminate 2	laminate 3	laminate 4
0.00	3.40	4.31	3.63	3.30
0.25	3.90	4.50	3.79	3.45
1.00	3.89	4.52	3.81	3.46

(*d*) *Poisson's ratio parallel to fibres in uncracked laminae*

crack density, δ	laminate 1	laminate 2	laminate 3	laminate 4
0.00	0.300	0.253	0.241	0.268
0.25	0.354	0.260	0.248	0.275
1.00	0.322	0.260	0.247	0.275

effective material properties, E_1, E_2, G_6 and ν_{12}, obtained from the cracked laminate analysis (Li *et al*. 1994) are in good agreement with those given by the damage representation, provided the damage parameter is properly defined and the damage-related material constants are properly determined. Therefore the replacement of the cracked lamina by a fictitious material is a reasonable approximation as far as this cracked lamina itself is concerned. However, the influence of the damage in the cracked lamina on its surrounding laminae still needs to be examined. In other words, how much do the interactions between the cracked lamina and its neighbouring uncracked lamina affect the behaviour of the uncracked lamina in terms of its effective material properties?

The effective properties of the uncracked laminae (0°-ply) in four of the above-mentioned cracked laminates have been calculated in the same way as those of the cracked laminae and the results are listed in table 7. It can be seen that these uncracked laminae are only slightly affected by the presence of the damage in the neighbouring lamina, and the differences between the original and the effective material properties are negligible. In other words, ignoring the changes in the effective properties of the material in uncracked laminae resulting from the interactions between the uncracked lamina and its neighbouring cracked lamina, as proposed in the damage model, does not introduce significant errors.

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4. Laminate analysis using the effective material properties obtained from the damage representation

The damage evolution law will be developed in §§ 5 and 6 of this paper to provide a complete description of the cracking process in a laminate. This requires the damage representation to be applied so that a cracked laminate can be analysed using a conventional laminate theory, with the cracked lamina replaced by a fictitious homogeneous material with the effective material properties obtained from the damage representation with a defined level of damage. The laminate analysis used is based on classical laminate theory (CLT) and is independent of the micromechanical cracked laminate analysis (Li *et al.* 1994) except for the provision of some input data, converting crack density to the damage parameter and obtaining the parameter k for the laminate. Once the damage parameter is known, all the effective properties of the fictitious material which replaces the cracked lamina can be obtained from the damage representation, equations (2.11).

The stress–strain relation for the lamina, designated by the subscript ℓ, in its local coordinate system (aligned with the principal directions of the material), is

$$\sigma_\ell = Q_\ell \epsilon_\ell, \tag{4.1}$$

where the stress (tensor), σ_ℓ, and strain (tensor), ϵ_ℓ, involve only in-plane components since in CLT a plane-stress state is assumed. In equation (4.1),

$$Q_\ell = Q_\ell^0 \qquad \text{(if lamina } \ell \text{ is uncracked)}, \tag{4.2}$$

$$Q_\ell = Q_\ell(\omega_\ell) = Q_\ell^0 + Q'_\ell \omega_\ell \qquad \text{(if lamina } \ell \text{ is cracked)}, \tag{4.3}$$

$$\left.\begin{aligned} Q_\ell^0 &= \begin{bmatrix} \dfrac{E_1^0}{1-\nu_{12}^0\nu_{21}^0} & \dfrac{\nu_{12}^0 E_2^0}{1-\nu_{12}^0\nu_{21}^0} & 0 \\ \dfrac{\nu_{12}^0 E_2^0}{1-\nu_{12}^0\nu_{21}^0} & \dfrac{E_2^0}{1-\nu_{12}^0\nu_{21}^0} & 0 \\ 0 & 0 & G_6^0 \end{bmatrix}, \\ Q'_\ell &= \begin{bmatrix} -\dfrac{\nu_{12}^0\nu_{21}^0 E_1^0}{(1-\nu_{12}^0\nu_{21}^0)^2} & -\dfrac{\nu_{12}^0 E_2^0}{(1-\nu_{12}^0\nu_{21}^0)^2} & 0 \\ -\dfrac{\nu_{12}^0 E_2^0}{(1-\nu_{12}^0\nu_{21}^0)^2} & -\dfrac{E_2^0}{(1-\nu_{12}^0\nu_{21}^0)^2} & 0 \\ 0 & 0 & -kG_6^0 \end{bmatrix}. \end{aligned}\right\} \tag{4.4}$$

In obtaining equation (4.3) from the expressions for the effective material properties in equations (2.11), higher-order terms of the damage parameter ω have been neglected based on the assumption of small damage.

Denote the stress, strain and the stiffness (tensors) of lamina ℓ in the laminate's global coordinate system by τ_ℓ, γ_ℓ and q_ℓ, respectively. Using the coordinate transformations,

$$\tau_\ell = T_\ell \sigma_\ell, \quad \gamma_\ell = T_\ell^{-\mathrm{T}} \epsilon_\ell, \quad q_\ell = T_\ell Q_\ell T_\ell^{\mathrm{T}}, \tag{4.5}$$

where T_ℓ is the coordinate transformation matrix for lamina ℓ as given in equation (3.4),

$$T_\ell = T(\theta_\ell) \quad \text{and} \quad T_\ell^{-\mathrm{T}} = T^{\mathrm{T}}(-\theta_\ell), \tag{4.6}$$

and θ_ℓ is the orientation of the material axes relative to the loading axes of lamina ℓ, relation (4.1) can be transformed to the global coordinate system,

$$\tau_\ell = q_\ell \gamma_\ell, \tag{4.7}$$

where the damage is included via q_ℓ.

In CLT, strains at a distance z from the reference surface of the laminate are associated with the generalized strains e, as a consequence of Love–Kirchhoff hypothesis, by

$$\gamma = Le, \tag{4.8}$$

where

$$L = L(z) = \begin{bmatrix} 1 & 0 & 0 & z & 0 & 0 \\ 0 & 1 & 0 & 0 & z & 0 \\ 0 & 0 & 1 & 0 & 0 & z \end{bmatrix} \tag{4.9}$$

and

$$e = [e_1, e_2, e_3, e_4, e_5, e_6]^{\mathrm{T}}.$$

Multiplying equation (4.7) by L^{T} and then integrating it over the thickness of the laminate (assembling all the laminae into a laminate) results in

$$s = De, \tag{4.10}$$

where

$$s = [s_1, s_2, s_3, s_4, s_5, s_6]^{\mathrm{T}}$$

are the stress resultants of the laminate and

$$D = \sum_\ell \begin{bmatrix} (h_\ell - h_{\ell-1})q_\ell & \frac{1}{2}(h_\ell^2 - h_{\ell-1}^2)q_\ell \\ \frac{1}{2}(h_\ell^2 - h_{\ell-1}^2)q_\ell & \frac{1}{3}(h_\ell^3 - h_{\ell-1}^3)q_\ell \end{bmatrix}, \tag{4.11}$$

where q_ℓ is defined in equation (4.5) and $h_{\ell-1}$ and h_ℓ are the z-coordinates of the bottom and top surfaces of lamina ℓ.

Equation (4.10) governs the behaviour of the laminate in which cracked laminae have been replaced by fictitious materials of effective material properties defined according to the proposed damage representation. As established in the previous section, this laminate of fictitious laminae will respond to loads effectively the same as the real cracked laminate.

The laminate theory formulated above can provide many aspects of the behaviour of cracked laminates. However, in the literature, results are only available for the global effective properties of laminates. Comparisons will be made with these properties, namely Young's modulus, Poisson's ratio and shear modulus, defined, respectively, as

$$E_x = s_1/e_1, \qquad \text{when } s_2 = s_3 = s_4 = s_5 = s_6 = 0, \tag{4.12}$$

$$\nu_{xy} = -e_2/e_1, \qquad \text{when } s_2 = s_3 = s_4 = s_5 = s_6 = 0, \tag{4.13}$$

$$G_{xy} = s_6/e_6, \qquad \text{when } s_1 = s_2 = s_3 = s_4 = s_5 = 0. \tag{4.14}$$

Highsmith & Reifsnider (1982) presented a set of experimental data for a $[0^\circ/90^\circ_3]_{\mathrm{s}}$ glass/epoxy laminate including stiffness reduction versus crack density in the 90° layer. Theoretical predictions have been made by Hashin (1985) and Li *et al.* (1994) for the same laminate using different methods of micromechanical cracked laminate

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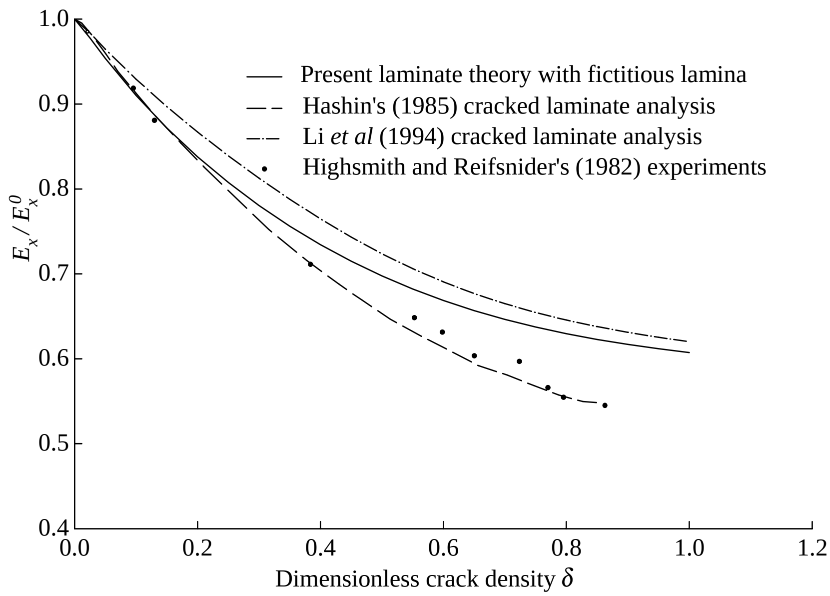

Figure 3. Relative effective Young's modulus of laminate 1 versus dimensionless crack density.

analysis. Hashin's (1985) prediction is a complementary energy-based approach and, therefore, gives a lower bound. Since his theory assumes that the stresses do not vary through the thickness of each lamina, the solution is not meant to be exact. In other words, this theoretical lower bound could be substantially lifted if more refined assumptions were introduced. The theoretical prediction in Li *et al*. (1994), on the other hand, is based on potential energy and it produces an upper bound. The same problem is re-examined here using classical laminate theory incorporating the damage representation given in equations (2.11). This solution is neither an upper bound nor a lower bound. In this analysis, the damage parameter ω, is calculated from the given crack density using the finite strip micromechanical analysis (Li *et al*. 1994). Apart from this, the present analysis is independent of the micromechanical analysis. The curve of the relative effective stiffness of the laminate, E_x, versus crack density resulting from the present analysis is shown in figure 3. The differences between the present study and the micromechanics analyses are due to the differences in the assumptions introduced. The most important difference is that in the present analysis the cracks have been smeared out and all the laminae are treated as continua with their effective properties given by equations (2.11), while in the micromechanical analyses cracks maintain their discrete nature. All of the solutions are within an acceptable range, given the nature of the problem.

Figure 4 shows the comparisons between the theoretical predictions using the present method and the experimental data obtained by Talreja *et al*. (1992) for the relative effective Poisson's ratio ν_{xy} of the laminate (with respect to the Poisson's ratio of the same laminate in an undamaged state) in the direction perpendicular to the cracks versus crack density. According to the theoretical analysis, there is not as much nonlinearity in the change in Poisson's ratio with respect to crack density as appeared in the experimental results of Talreja *et al*. in the early stages, i.e. $\delta < 0.5$. Therefore, the linear fit to the experimental data used by Talreja *et al*. (1992) stands

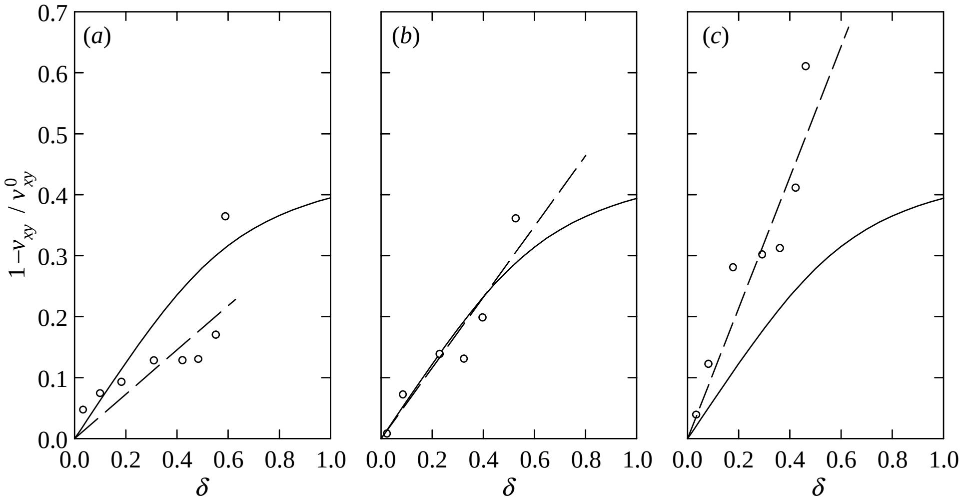

Figure 4. Change of relative effective Poisson's ratio of (*a*) laminate 2, (*b*) laminate 3, and (*c*) laminate 4 versus dimensionless crack density: ——, present laminate theory with fictitious lamina; ○ ○ ○, experimental results (Talreja *et al*. 1992); – – –, linear fit to experimental data (Talreja *et al*. 1992).

on reasonable ground. Comparing the theoretical results obtained here with each other, it can be seen that the predicted results for the three laminates are very close to each other because the elastic material properties of the three laminates used for the analysis are very close to each other. The present theory assumes that the overall behaviour of cracked laminates at a fixed crack density is predominantly elastic and is dominated by the elastic properties of the materials.

However, in figure 4, only (*b*) shows reasonable agreement with the experimental results, while in (*a*) and (*c*) significant discrepancies are present. The experimental results for the three laminates in Talreja *et al*. (1992) show very pronounced differences between them. According to Talreja *et al*. (1992) and Talreja (1995), they are due to the different extent of blunting occurring at the crack tips in these laminates associated with the material toughness since the differences in the materials involved in the three laminates lie mainly in their toughnesses. While this is still a possibility yet to be confirmed, the present theory has clearly not taken account of this effect. This comparison is discussed further in § 7 *a*.

Hashin (1985) presented a case involving the same laminate as laminate 1 in table 1 and predicted the effective in-plane shear modulus, G_{xy}, of the laminate relative to that of the laminate in the undamaged state, G_{xy}^0, as a function of crack density. This case was also analysed, using their cracked laminate analysis, by Li *et al*. (1994). Similar results are obtained using the proposed 'fictitious' laminate analysis and plotted in figure 5 where the two curves shown for the present analysis correspond to the two k values given in table 5, respectively. They are both in reasonable agreement with Hashin's results. As expected, the curve corresponding to the first k value (0.678) produces better agreement at low crack densities where parameter k is indeed constant. As yet, no experimental results regarding shear behaviour are available to the authors for comparison.

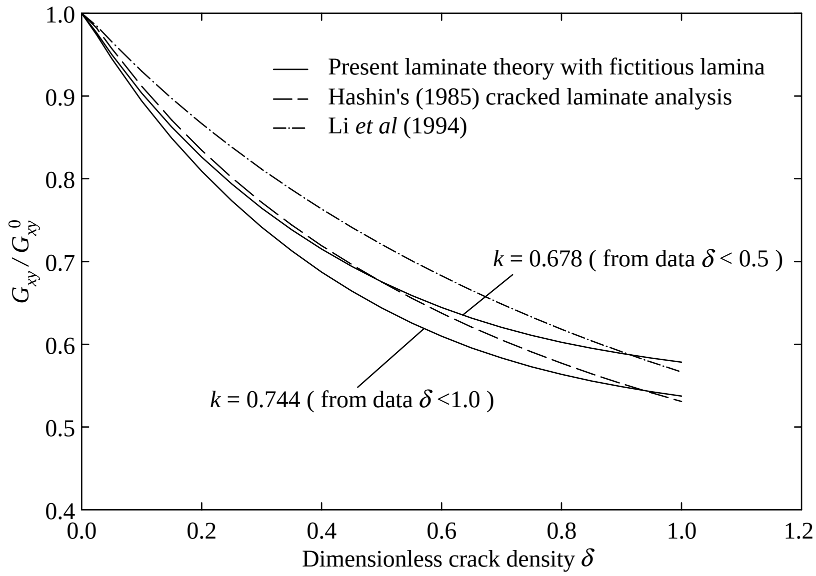

Figure 5. Relative shear modulus of laminate 1 versus dimensionless crack density.

5. Incremental constitutive relation for a laminate

Damage growth is the next aspect to be pursued, this being in the particular form of crack multiplication as noted in § 1. When the damage is described by a vector variable, the vector maintains its orientation and, therefore, it can be fully defined by its length which is represented by the damage parameter ω. A damage growth law is a description of the development of this parameter which is determined by the load and the loading history. In general, this presents a nonlinear problem due to the degradation in material properties. Conventional solution techniques for nonlinear problems are usually based on an incremental approach and this will be followed here. In order to incorporate this incremental constitutive relation into the laminate analysis described in the last section, it will be expressed in a form suitable for this type of application and expressed in terms of an incremental relation between the stress resultants and generalized strains.

The incremental form of the stress, strain and damage relation for lamina ℓ can be obtained from equation (4.7) as

$$\mathrm{d}\tau_\ell = \frac{\partial \tau_\ell}{\partial \gamma_\ell}\,\mathrm{d}\gamma_\ell + \frac{\partial \tau_\ell}{\partial \omega_\ell}\,\mathrm{d}\omega_\ell = q_\ell\,\mathrm{d}\gamma_\ell + q'_\ell \gamma_\ell\,\mathrm{d}\omega_\ell, \tag{5.1}$$

where

$$q'_\ell = T_\ell Q'_\ell T_\ell^{\mathrm{T}}, \tag{5.2}$$

and, for the sake of convenience and also maintaining an acceptable level of approximation, γ_ℓ will be taken as the average strain in the lamina. This is obtained from equations (4.8) and (4.9) as

$$\gamma_\ell = L_\ell e \qquad \text{with } L_\ell = L(\tfrac{1}{2}(H_{\ell-1} + h_\ell)). \tag{5.3}$$

Integration of equation (5.1) is performed over the thickness of the laminate in a manner similar to that used in obtaining equation (4.10), resulting in

$$\mathrm{d}s = D\,\mathrm{d}e + \sum_{\ell} D'_{\ell} e\,\mathrm{d}\omega_{\ell}, \tag{5.4}$$

where de and ds are the incremental generalized strains and the stress resultants of the laminate and

$$D'_{\ell} = \begin{bmatrix} (h_{\ell} - h_{\ell-1})q'_{\ell} & \frac{1}{2}(h_{\ell}^2 - h_{\ell-1}^2)q'_{\ell} \\ \frac{1}{2}(h_{\ell}^2 - h_{\ell-1}^2)q'_{\ell} & \frac{1}{3}(h_{\ell}^3 - h_{\ell-1}^3)q'_{\ell} \end{bmatrix}. \tag{5.5}$$

D has been given in equation (4.11). In the laminate, only cracked laminae contribute to the second term on the right-hand side of equation (5.4) and therefore the summation is over all the cracked laminae in the laminate.

Equation (5.4) relates the incremental stress resultants to the incremental generalized strains, but involved in the relation are the incremental damage parameters of all the cracked laminae which cannot be determined in equation (5.4). Extra information is required and the aim is to eliminate these damage-parameter increments from equation (5.4). This will be pursued in the next section by making use of the concept of a damage surface.

6. Damage surface and damage growth in a laminate

Microscopic examinations of cracked laminates show that most cracks span the full thickness of the lamina in which they occur, at least, when they become fully developed, and are arrested by the interfaces between the lamina and the adjacent ones. This suggests that the propagation of cracks through the thickness of the lamina, which is likely to be an unstable process, is not a dominant feature in the overall behaviour of a laminate undergoing transverse matrix cracking and, therefore, can be ignored. A similar assumption has been made in Nairn (1989) using a similar argument. This leads to the simplification that the behaviour of the material bounded by two cracks in the cracked lamina is determined by the average stresses over the thickness of the lamina, to be referred to hereafter as the stresses in the lamina. Thus, the damage surface f for lamina ℓ can be assumed to be defined in terms of the stresses σ_{ℓ} in the lamina. In general, the damage surface for lamina ℓ can be expressed in the form

$$f(\sigma_{\ell}, \omega_{\ell}) = 1. \tag{6.1}$$

The inclusion of ω_{ℓ} in f reflects the 'strengthening' effect of the material of lamina ℓ due to the existence of cracks, i.e. the tendency to require higher local stresses to increase the damage level. This effect is associated with the influence of defects in the material and is often referred to as the size effect (Manders *et al*. 1983; Wisnom 1991). The strength of virgin material is usually reduced by the defects in the material, some of which initiate the damage. With the increasing development of damage, more and more defects develop into damage. The number and severity of defects decrease in the undamaged part of the material and, therefore, the local strength increases. Results associated with size effects are often based on Weibull analysis (Weibull 1951) and are formulated for cases which usually involve only uniaxial stress states. In the absence of sufficient information about the size effects of the material, it is assumed here that f is defined in the form

$$f(\sigma_\ell, \omega_\ell) = (1 + h_\ell \omega_\ell^{\eta_\ell})^{-1} F(\sigma_\ell) = 1, \tag{6.2}$$

where F is a function of σ_ℓ taken from one of the conventional failure criteria and h_ℓ and η_ℓ are properties of the material of lamina ℓ associated with the size effects of the material. In the case of uniaxial stress states, h_ℓ and η_ℓ are the Weibull parameters (Weibull 1951).

From equation (6.2) the initiation of damage is completely determined by the employed failure criterion described by F. As the damage develops the damage surface expands, as can be seen more easily if the damage surface equation (6.2) is rewritten as

$$F(\sigma_\ell) = 1 + h_\ell \omega_\ell^{\eta_\ell}. \tag{6.3}$$

While crude, this is convenient for mathematical manipulation and is simple because only one damage mechanism, transverse matrix cracking, is included. Any refinements to the definition of the damage surface would have to be based on experimental data which are unavailable yet for many other damage mechanisms.

An infinitesimal change of the damage state in lamina ℓ as a result of an infinitesimal change in the stress resultants applied to the laminate, requires the satisfaction of the following equation so that the internal state of stress, strain and damage remains on the damage surface,

$$\frac{\partial F}{\partial \sigma_\ell}\frac{\partial \sigma_\ell}{\partial s}\,\mathrm{d}s + \frac{\partial F}{\partial \sigma_\ell}\sum_j \frac{\partial \sigma_\ell}{\partial \omega_j}\,\mathrm{d}\omega_j - h_\ell \eta_\ell \omega_\ell^{\eta_\ell - 1}\,\mathrm{d}\omega_\ell = 0, \tag{6.4}$$

where j covers every cracked lamina reflecting the influence of the damage in lamina j on the internal state in lamina ℓ. Neglected here is the explicit interaction associated with individual cracks in two neighbouring cracked laminae which cross. This is similar to the situation where the local effects of crack tips at lamina interfaces are neglected and is justified by the same assumption that a lamina, either cracked or not, is under a plane-stress state effectively as discussed at the beginning of § 3. Approximations such as this represent the constraints of a continuum approach which cannot include such discrete effects. In contrast, discrete approaches such as fracture mechanics-based models are incapable of dealing with information of a distributed nature, e.g. the effects of a massive number of cracks distributed through a piece of material.

Substituting equation (5.4) into equation (6.4) and separating the terms for lamina ℓ from the others, one obtains

$$\left[-h_\ell \eta_\ell \omega_\ell^{\eta_\ell - 1} + \frac{\partial F}{\partial \sigma_\ell}\left(\frac{\partial \sigma_\ell}{\partial \omega_\ell} + \frac{\partial \sigma_\ell}{\partial s} D'_\ell e\right)\right]\mathrm{d}\omega_\ell + \frac{\partial F}{\partial \sigma_\ell}\sum_{j \neq \ell}\left(\frac{\partial \sigma_\ell}{\partial \omega_j} + \frac{\partial \sigma_\ell}{\partial s} D'_j e\right)\mathrm{d}\omega_j = -\frac{\partial F}{\partial \sigma_\ell}\frac{\partial \sigma_\ell}{\partial s} D\,\mathrm{d}e. \tag{6.5}$$

If the same equation as above is written for each of the cracked laminae, a set of simultaneous equations for all the incremental damage parameters in all the cracked laminae is obtained. The coefficient matrix may seem to be a full matrix, or in other

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words, $d\omega_\ell$ and $d\omega_j$ are coupled ($j \neq \ell$). This coupling can be eliminated and all the off-diagonal elements in the coefficient matrix vanish as shown below.

Stresses σ_ℓ, which are related to τ_ℓ by equation (4.5), depend on the applied stress resultants, s, and the damage parameter ω_j of all the cracked laminae in the laminate, i.e.

$$\sigma_\ell = \sigma_\ell(s, \omega_j) = Q_\ell \epsilon_\ell = Q_\ell T_\ell^{\mathrm{T}} \gamma_\ell = Q_\ell T_\ell^{\mathrm{T}} L_\ell e = Q_\ell T_\ell^{\mathrm{T}} L_\ell D^{-1} s. \tag{6.6}$$

From this, the following can be obtained:

$$\frac{\partial \sigma_\ell}{\partial s} = \frac{\partial \sigma_\ell}{\partial \epsilon_\ell} \frac{\partial \epsilon_\ell}{\partial \gamma_\ell} \frac{\partial \gamma_\ell}{\partial e} \frac{\partial e}{\partial s} = Q_\ell T_\ell^{\mathrm{T}} L_\ell D^{-1}, \tag{6.7}$$

$$\begin{aligned} \frac{\partial \sigma_\ell}{\partial \omega_\ell} &= \frac{\partial Q_\ell}{\partial \omega_\ell} T_\ell^{\mathrm{T}} L_\ell D^{-1} s + Q_\ell T_\ell^{\mathrm{T}} L_\ell \frac{\partial D^{-1}}{\partial \omega_\ell} s \\ &= (Q'_\ell T_\ell^{\mathrm{T}} L_\ell - Q_\ell T_\ell^{\mathrm{T}} L_\ell D^{-1} D'_\ell) e \end{aligned} \tag{6.8}$$

and

$$\frac{\partial \sigma_\ell}{\partial \omega_j} = -Q_\ell T_\ell^{\mathrm{T}} L_\ell D^{-1} D'_j e \qquad (j \neq \ell). \tag{6.9}$$

In obtaining the second part of the above expression use has been made of the following identities:

$$\frac{\partial (D^{-1})}{\partial \omega_i} = -D^{-1} \frac{\partial D}{\partial \omega_i} D^{-1} = -D^{-1} D'_i D^{-1} \quad \text{and} \quad e = D^{-1} s. \tag{6.10}$$

Substituting equations (6.7)–(6.9) into equation (6.5), the coupling terms (those inside the summation sign) cancel each other.

The incremental damage parameters of lamina ℓ can be obtained in terms of incremental generalized strain as

$$d\omega_\ell = B_\ell \, de, \tag{6.11}$$

where

$$B_\ell = \frac{\dfrac{\partial F}{\partial \sigma_\ell} Q_\ell T_\ell^{\mathrm{T}} L_\ell}{h_\ell \eta_\ell \omega_\ell^{\eta_\ell - 1} - \dfrac{\partial F}{\partial \sigma_\ell} Q'_\ell T_\ell^{\mathrm{T}} L_\ell e}. \tag{6.12}$$

The incremental stress-resultant-generalized strain relation, which has taken account of the growth of damage in all the laminae of the laminate, can then be obtained by substituting equation (6.11) into equation (5.4) as

$$ds = D_d \, de, \tag{6.13}$$

where

$$D_d = D + \sum_\ell D'_\ell e B_\ell. \tag{6.14}$$

Equation (6.13) looks similar to the stress-resultant-generalized strain relation equation (4.8) and indeed it is the incremental form of that relation. Built into it is the influence of damage growth of laminae experiencing transverse matrix cracking.

Matrix D_d gives the tangent stiffness of the laminate at the current deformation and damage state. It can be used in an incremental laminate analysis involving transverse matrix cracking. In such an analysis, when the stress-resultant increment ds is applied, the generalized strain increment de can be determined from equation (6.13) as

$$\mathrm{d}e = D_d^{-1}\,\mathrm{d}s. \tag{6.15}$$

Substituting this back into equation (6.11), $\mathrm{d}\omega_\ell$ can be obtained once de is known.

It should be pointed out that a negative value produced by equation (6.11) for $\mathrm{d}\omega_\ell$ indicates unloading in lamina ℓ and equation (6.11) should be replaced by

$$\mathrm{d}\omega_\ell = 0, \tag{6.16}$$

because of the irreversible nature of damage. It means that unloading does not follow the same path as that of loading. This behaviour provides the mechanism in the theoretical model which allows energy dissipation as a result of damage development in the material. When unloading is identified in a lamina, the lamina should be excluded from the summation in equation (6.14). This operation discounts the contribution of the damage growth in this particular lamina but not the existing damage in it. When the damage in a lamina stops growing, the damage state stays fixed and the effects of fixed damage have been included in the other part of the right-hand side expression, D, of equation (6.14).

7. Examples

The model for the damage in the form of transverse matrix cracking in laminated composites has been developed in the previous sections. Its applicability is not subject to any restriction from the layup of the laminate, such as symmetric or asymmetric, balanced or not, cross-ply or non-cross-ply. In other words, it applies to laminates of unidirectional laminae with arbitrary construction, provided that the mode of damage is in the form of transverse matrix cracking. It is also capable of dealing with any combination of loads expressed in terms of stress resultants. However, constrained by the availability of published experimental data, only two cases will be studied here: a cross-ply laminate under uniaxial tension; and an angle-ply laminate under biaxial tension. When defining the damage surface in the analysis, the maximum stress criterion (Tsai & Hahn 1980) is employed to define the function F in equation (6.2) for the sake of simplicity but there is no restriction in the theory on using any other criterion. The strengthening behaviour due to size effects, though being recognized in other treatments of damage (e.g. Nairn 1989; Laws & Dvorak 1988), has not been included in other solutions using continuum damage mechanics. Consequently, the parameters h and η are not available. For the purpose of illustration, two sets of values, $h = \eta = 0$, corresponding to the case where no size effect is present, and $h = \eta = 0.5$ have been chosen on a somewhat arbitrary basis.

(*a*) *A cross-ply laminate subjected to uniaxial tension*

A glass/epoxy cross-ply laminate, $[0°/90°_3]_\mathrm{s}$, was tested by Highsmith & Reifsnider (1982). It is one of the most frequently cited cases in the literature (Hashin 1985; Talreja 1985*b*), and many models for damaged composites have been compared with the data from this experiment. Most of these models are fracture-mechanics-based

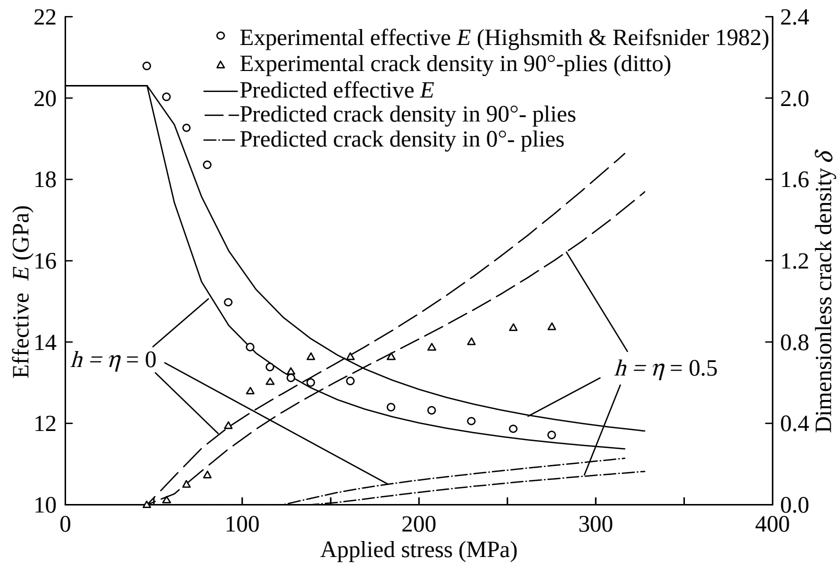

Figure 6. Effective Young's modulus and crack density versus applied stress.

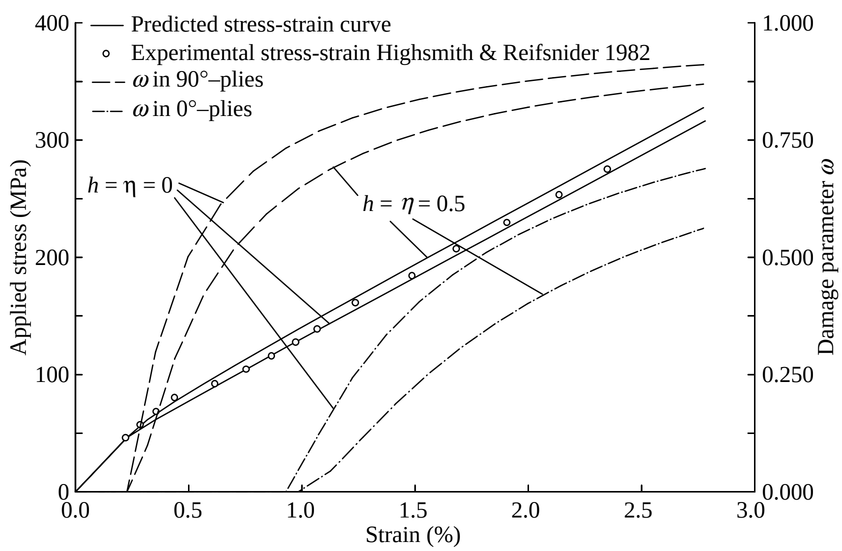

Figure 7. Applied stress and the damage parameter versus strain.

and are restricted to cross-ply laminates, unlike the present model (see § 7 *b*). In these previous analyses, predictions of the properties and behaviour of the laminate were made when the crack density was prescribed and, to the author's knowledge, theoretical stress–strain curves have not been reproduced independently of experiments. An example of an earlier study which does include stress–strain curves is the paper by Eckold *et al.* (1995). However, these curves were produced by an iterative procedure which matches the output of a micromechanics analysis to experimental stress–strain curves. From this process emerges a value for the mean cracking energy (a supposed material property), though the universal relevance of this parameter

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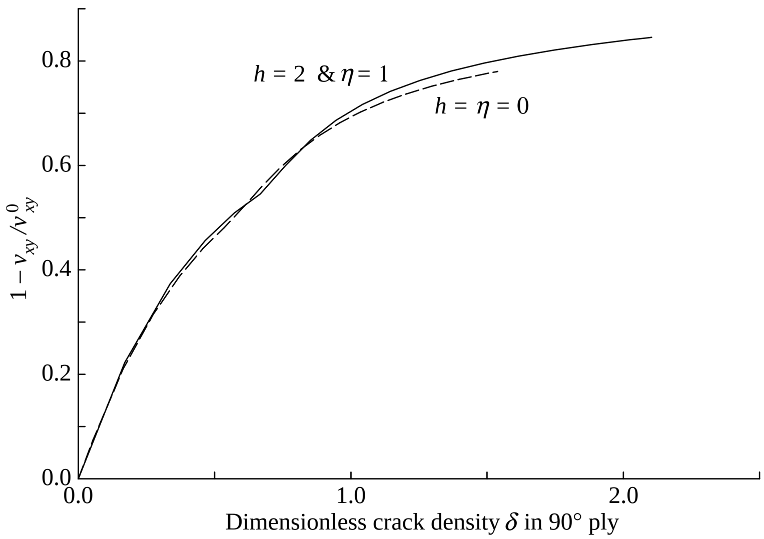

Figure 8. Relative change in Poisson's ratio of the laminate versus crack density.

is unclear. This approach, while interesting and constructive, does not represent a method for predicting laminate stress–strain curves *ab initio*. With the damage growth model presented above this is now possible. The elastic material constants for the laminae were given in Hashin (1985) and Highsmith & Reifsnider (1982): see laminate 1 in table 1. In order to construct the damage surface, the strength properties are also required but these are not provided in the literature. They are defined here by interpolating the strength properties, assuming them to be proportional to the ratio of the corresponding elastic constants of a similar material given as laminate 5 (Li *et al*. 1993) in table 1. The predicted results for the stiffness of the laminate and the crack density in the cracked lamina versus applied stress (averaged over the thickness of the laminate) curves are shown in figure 6. The two sets of curves corresponding to $h = \eta = 0$ and $h = \eta = 0.5$ each give reasonable predictions for the stiffness reduction in different parts of the curves, the former at higher stress levels and the latter at lower stresses. The predicted crack densities for these two sets of parameters seem to embrace the experimental values reasonably well. In general, with the size effect included, the laminate shows higher stiffness and slower crack density growth than without it. This illustrates the strengthening effect in a straightforward manner. Figure 7 shows the stress–strain curves, which are in excellent agreement with the experimental data. Also included in figure 7 are the predictions of the damage parameters in both the 90° and 0° laminae.

It is interesting to note that the 0° plies on both sides of the laminate start to crack at a stress of about 125 MPa. This phenomenon was not reported in Highsmith & Reifsnider (1982) but it is plausible that it occurs as the result of the Poisson's ratio effect. The width of the specimen would affect the result greatly due to the free-edge effects. Further experiments are encouraged in order to verify this.

The importance of cracking in the 0° ply lies in its effects on the overall Poisson's ratio of the laminate as shown in figure 8. It results in a fluctuation in the change (less obvious in the one corresponding to $h = \eta = 0.5$) of the parameter which corresponds to the initiation of cracking in the 0° plies. It is suspected that the

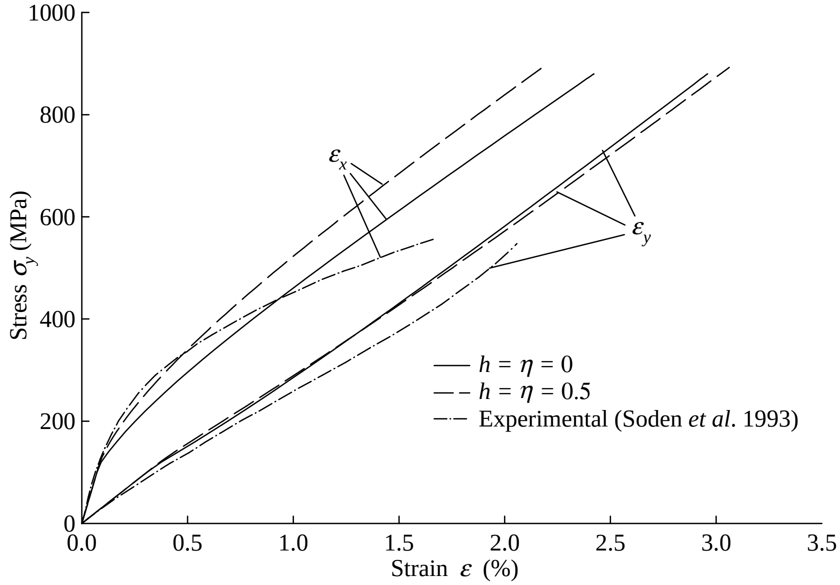

Figure 9. Theoretical and experimental stress–strain curves of a ±55° laminate under biaxial tension at a ratio 2:1.

nonlinear variations in the experimentally measured Poisson's ratio in Talreja *et al*. (1992) may be the result of this. Should cracking in 0° plies take place, the validity of extracting the properties of cracked laminates from experimental data assuming that only the 90° plies were cracked becomes questionable. Unfortunately, predictions for the cases described in Talreja *et al*. (1992) using the present theory cannot be made because the strength properties of the materials are unavailable (R. Talreja and S. Yalvac, personal communication).

(*b*) *An angle-ply laminate subjected to 2:1 ratio biaxial tensile loads*

A glass/epoxy [+55°/−55°/+55°/−55°] angle-ply laminate is analysed as a second example. The laminate is subjected to biaxial tensile loads applied in a 2:1 ratio. This is the basic loading condition for a cylindrical pressure vessel. Experimental stress–strain data for the internal pressurization of a closed-ended cylinder with this wall construction having an internal diameter of 100 mm and a nominal wall-thickness of 1 mm are given by Soden *et al*. (1993) and the appropriate material properties are those listed in table 1 under laminate 5. Because the laminate layup is antisymmetric, there would be some twisting as a result of in-plane tension if the laminate were free to deform. To simulate a cylindrical pressure vessel, such twisting is suppressed in the analysis.

The stress σ_y ($\sigma_y : \sigma_x = 2 : 1$) averaged over the thickness of the laminate is plotted against the strains ϵ_x and ϵ_y in figure 9, respectively. It can be seen that ϵ_y is not too sensitive to size effects because the behaviour in the y-direction of the laminate is fibre dominated and is not sensitive to matrix cracking. ϵ_x is much more sensitive to size effects since, in this direction, the matrix in the material plays a significant role. The comparison with the experimental data (Soden *et al*. 1993) is encouraging. Without the damage model, a linear prediction would produce straight lines as extensions of

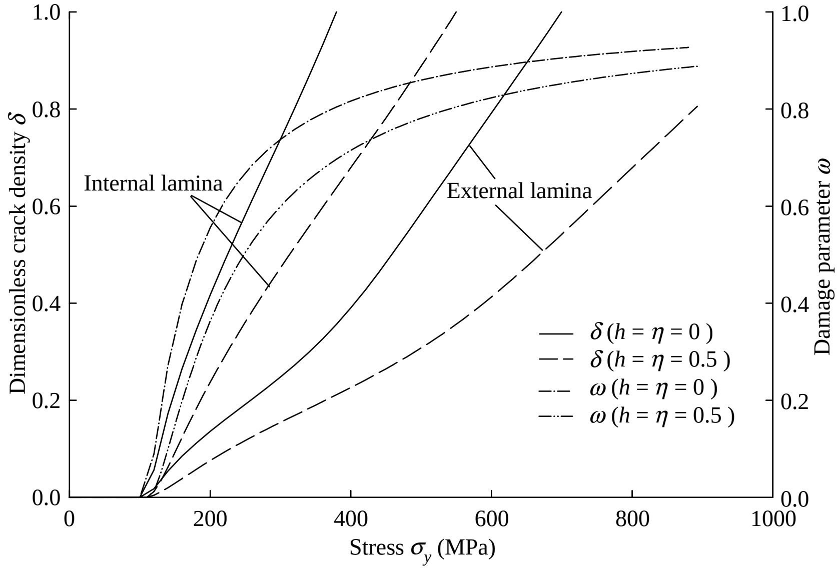

Figure 10. Predicted crack density and damage parameter versus applied load for a ±55° laminate under biaxial tension at a ratio 2:1.

the initial linear segments for both ϵ_x and ϵ_y. Neither crack density data nor any other damage measure are available from the experiments. Therefore comparisons between experiments and theory are not possible. However, the predictions of various aspects of the damage are given in figure 10. The predicted crack density in each lamina has been non-dimensionalized with respect to the thickness of the lamina. As has been shown in the previous example, the crack density is, in general, sensitive to size effects. Due to the angle-ply layup of the laminate and the loading, all the laminae behave in the same way in terms of the stresses and strains referred to the local material axes and the damage parameter. It is interesting to note that the calculated crack density in an external lamina is significantly different from that in an internal lamina. This is a consequence of the different constraints to which an external lamina and an internal lamina are subjected. A crack reduces stresses to a greater extent in an external lamina than in an internal one.

8. Conclusions

A damage representation for cracked laminates appropriate to the particular mechanism of damage by transverse matrix cracking in the direction parallel to the fibres has been proposed. It stems from a rearrangement of Talreja's formulation, which is believed to give clearer physical meaning to the material constants introduced by the damage considerations. With the help of the assumption that the interlaminar force interactions in a cracked laminate do not make significant differences as far as the effective material properties of the laminae (whether cracked or not) are concerned, and hence can be neglected, most of the constants can be determined immediately, leaving only one layup-dependent damage-related material constant associated with in-plane shear. It has been shown that this damage-related material constant can be estimated theoretically. This approach therefore brings significant simplifications to the damage representation and, at the same time, minimizes the dependence of the theory on experiments for determining the damage-related material constants. The assumption introduced has been shown to be reasonable based on the results from a micromechanical cracked laminate analysis which is independent of the damage representation. It has been established that the overall behaviour of a cracked laminate can be satisfactorily predicted by a laminate theory, with the cracked lamina replaced by a fictitious material with the effective material properties obtained from the damage representation at the prescribed levels of damage. Thus, in conjunction with a damage growth law which describes the evolution of the damage parameter ω, it has been developed into a complete damage model which is able to predict the consequences of the cracking process for practical cases.

The damage growth law proposed is formulated for the process of multiplication of transverse matrix cracks. By making use of the concept of a damage surface, an incremental constitutive relation between the stress resultants and the generalized strains has been obtained which is convenient for applications in laminate analysis. The examples analysed have shown good agreement with experimental results in spite of the lack of detailed material behaviour data.

Unlike many existing damage models, the one presented in this paper is applicable to arbitrary laminates and is not restricted to cross-ply layups. General loading conditions can be treated, provided the damage is in the form of transverse matrix cracking.

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References

Allen, D. H., Harris, C. E. & Groves, S. E. 1987*a* A theoretical constitutive theory for elastic composites with distributed damage. I. Theoretical development. *Int. J. Solids Struct.* **23**, 1301–1318.

Allen, D. H., Harris, C. E. & Groves, S. E. 1987*b* A theoretical constitutive theory for elastic composites with distributed damage. II. Application to matrix cracking in laminated composites. *Int. J. Solids Struct.* **23**, 1319–1338.

Altus, E. & Ishai, O. 1986 Transverse cracking and delamination interaction in the failure process of composite laminates. *Comp. Sci. Tech.* **26**, 59–77.

Coleman, B. D. & Noll, W. 1963 The thermodynamics of elastic materials with heat conduction and viscosity. *Arch. Ration. Mech. Analysis* **13**, 167–178.

Eckold, G. C., Hancox, N. L. & Lee, R. J. 1995 Application of micromechanics in the prediction of damage initiation and growth in structural composites. In *Proc. 3rd Conf. Deformation and Fracture of Composites, University of Surrey, Guildford, UK, 27–29 March 1995*, pp. 66–76.

Fan, J. & Zhang, J. 1993 *In situ* damage evolution and micro/macro transition for laminated composites. *Comp. Sci. Tech.* **47**, 107–118.

Garrett, K. W. & Bailey, J. E. 1977 Multiple transverse fracture in 90° cross-ply laminates of a glass fibre-reinforced polyester. *J. Mater. Sci.* **12**, 157–168.

Hashin, Z. 1985 Analysis of cracked laminates: a variational approach. *Mech. Mater.* **4**, 121–136.

Herakovich, C. T., Aboundi, J., Lee, S. W. & Strauss, E. A. 1988 Damage in composite laminates: effects of transverse cracks. *Mech. Mater.* **7**, 91–107.

Highsmith, A. L. & Reifsnider, K. L. 1982 Stiffness-reduction mechanisms in composite laminates. In *Damage in composite materials* ASTM-STP-775 (ed. K. L. Reifsnider), pp. 103–117. American Society for Testing of Materials.
Krajcinovic, D. 1984 Continuum damage mechanics. *Appl. Mech. Rev.* **3**, 1–6.
Krajcinovic, D. & Fonseka, G. U. 1981 The continuum damage mechanics of brittle materials. I and II. *J. Appl. Mech.* **48**, 809–824.
Laws, N. & Dvorak, G. J. 1988 Progressive transverse cracking in composite laminates. *J. Composite Mater.* **22**, 900–916.
Lewinski, T. & Telega, J. J. 1996*a* Stiffness loss of laminates with aligned intralaminar cracks. I. *Arch. Mech.* **48**, 245–264.
Lewinski, T. & Telega, J. J. 1996*b* Stiffness loss of laminates with aligned intralaminar cracks. II. *Arch. Mech.* **48**, 265–280.
Li, S., Soden, P. D., Reid, S. R. & Hinton, M. J. 1993 Indentation of laminated filament-wound composite tubes. *Composites* **24**, 407–421.
Li, S., Reid, S. R. & Soden, P. D. 1994 A finite strip analysis of cracked laminates. *Mech. Mater.* **18**, 289–311.
McCartney, L. N. 1990 A three-dimensional stress-transfer model for cross-ply laminates of finite width containing transverse cracks. In *Proc. IMechE. 4th Int. Conf. FRC*, vol. 90, pp. 19–26.
Manders, P. W., Chou, T.-W., Jones, F. R. & Rock, J. W. 1983 Statistical analysis of multiple fracture in $0°/90°/0°$ glass fibre/epoxy resin laminates. *J. Mater. Sci.* **18**, 2876–2889.
Nairn, J. A. 1989 The strain release rate of composite microcracking: a variational approach. *J. Composite Mater.* **23**, 1106–1129.
Ogin, S. L. & Smith, P. A. 1987 A model for matrix cracking in crossply laminates. *ESA Jl* **11**, 45–60.
Pipkin, A. C. & Rivlin, R. S. 1959 The foundation of constitutive equations in continuum physics. *Arch. Ration. Mech. Analysis* **4**, 129–144.
Soden, P. D., Kitching, R., Tse, P. C. & Tsavalas, Y. 1993 Influence of winding angle on the strength and deformation of filament-wound composite tubes subjected to uniaxial and biaxial loads. *Composites Sci. Tech.* **46**, 363–378.
Sokolnikoff, I. S. 1956 *Mathematical theory of elasticity.* New York: McGraw-Hill.
Swanson, S. R. 1989 On the mechanics of microcracking in fibre composite laminates under combined stress. *J. Engng Mater. Tech.* **111**, 145–149.
Talreja, R. 1985*a* A continuum mechanics characterization of damage in composite materials. *Proc. R. Soc. Lond.* A **339**, 195–216.
Talreja, R. 1985*b* Transverse cracking and stiffness reduction in composite laminates. *J. Composite Mater.* **19**, 355–375.
Talreja, R. 1986 Stiffness properties of composite laminates with matrix cracking and interior delamination. *Eng. Fract. Mech.* **25**, 751–762.
Talreja, R. 1995 Damage mechanics concepts for durability analysis of composite materials. In *Proc. 3rd Int. Conf. on Deformation and Fracture of Composites, University of Surrey, Guildford, UK, 27–29 March 1995*, pp. 15–25.
Talreja, R., Yalvac, S., Yats, L. D. & Wetters, D. G. 1992 Transverse cracking and stiffness reduction in cross ply laminates of different matrix toughness. *J. Composite Mater.* **26**, 1644–1663.
Thionnet, A. & Renard, J. 1993 Meso–macro approach to transverse cracking in laminated composites using Talreja's model. *Composites Engng* **3**, 851–871.
Tsai, S. W. & Hahn, H. T. 1980 *Introduction to composite materials.* Westport, CT: Technomic.
Weibull, W. 1951 A statistical distribution function of wide applicability. *J. Appl. Mech.* **18**, 293–297.
Wisnom, M. R. 1991 Relationship between strength variability and size effect in unidirectional carbon fibre/epoxy. *Composites* **22**, 47–52.